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Some integrals of the Dedekind η-function

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ABSTRACT

This note presents selected values of the class of integrals

$$\int_0^\infty f(x)\eta^n(ix)\,dx.$$

The motivation for this study was to investigate integrals similar to one that Ramanujan included as item IV (5) in his famous 1913 letter to Hardy [1] containing the expression $\cosh(x)+\cos(x)$ in the denominator. Actually, the first to examine an integral of this class appears to have been J.W.L. Glaisher [2] in 1871; it seems doubtful, however, that Ramanujan would have been aware of this work. The study of these integrals quickly led to the problem of evaluating integrals of the Dedekind-eta function, and particularly pertaining to the class

$$\int_0^\infty f(x)\eta^n(ix)\,dx. \tag{1}$$

This is probably to have been expected, since Ramanujan devotes a good deal of space in his notebooks to such integrals. These and many others can be found in Berndt's seminal exposition of Ramanujan's work [3]. Other Dedekind function integrals appear scattered in the literature, such as in references [4,5] and works cited there. The aim of this paper is to present a number of results of the form (1) having a more elementary character.

Note that for the real nome $q=e^{-2\pi x}$, our instance of the Dedekind η-function is given by

$$\eta(ix)=q^{1/24}\prod_{n=1}^\infty\left(1-q^n\right). \tag{2}$$

We begin with a simple rearrangement of Euler's identity [6]

$$\eta(ix)=\frac{2}{\sqrt{3}}\sum_{n=0}^\infty\cos\bigl[(2n+1)\pi/6\bigr]q^{(2n+1)^2/24}. \tag{3}$$

Then, after a change of variable $q=e^{-2\pi x}$, one has

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$$\int_0^1 \frac{dq}{q} q^y \eta(ix) = \frac{2\pi}{\sqrt{3}} \sum_{n=0}^{\infty} \cos\left[(2n+1)\frac{\pi}{6}\right] \int_0^{\infty} dx\, e^{-2\pi[y+(2n+1)^2/24]x}$$

$$= \frac{48\pi}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)\pi/6]}{(2n+1)^2+24y} = \pi\sqrt{\frac{2}{y}} \frac{\sinh\pi\sqrt{8y/3}}{\cosh\pi\sqrt{6y}}. \tag{4}$$

We therefore have the Laplace transform, where $t = 3\pi y$,

$$\int_0^{\infty} e^{-xt}\eta(ix)\,dx = \sqrt{\frac{\pi}{t}} \frac{\sinh 2\sqrt{\pi t/3}}{\cosh\sqrt{3\pi t}}. \tag{5}$$

In what follows, we shall assume that all parameters take on real values for which the integrals converge absolutely.

Let $F(t)$ denote the inverse Laplace transform of $f(x)$. Then from (5) by multiplying both sides by $F(t)$ and integrating over t, we have, by invoking Fubini's theorem, the identity

$$\int_0^{\infty} f(x)\eta(ix)\,dx = \sqrt{\pi}\int_0^{\infty} \frac{dt}{\sqrt{t}} F(t) \frac{\sinh 2\sqrt{\pi t/3}}{\cosh\sqrt{3\pi t}}. \tag{6}$$

We obtain in this way [8]

$$\int_0^{\infty} x^{-s}\eta(ix)\,dx = \frac{8\sqrt{3}\pi}{16^s(3\pi)^s} \frac{\Gamma(2s-1)}{\Gamma(s)}\left[\zeta\left(2s-1, \frac{1}{12}\right) + \zeta\left(2s-1, \frac{11}{12}\right)\right.$$

$$\left. - \zeta\left(2s-1, \frac{5}{12}\right) - \zeta\left(2s-1, \frac{7}{12}\right)\right] \quad (s>0). \tag{7}$$

Next, in (5) replace t by it and formally take the real part of both sides to get

$$\int_0^{\infty} \cos(xy)\eta(ix)\,dx = \sqrt{\frac{\pi}{2y}} \frac{\sinh\sqrt{8\pi y/3} + \sin\sqrt{8\pi y/3}}{\cosh\sqrt{8\pi y/3} + \cos\sqrt{8\pi y/3}}. \tag{8}$$

For $y \to 0$, (8) becomes

$$\int_0^{\infty} \eta(ix)\,dx = \frac{2\pi}{\sqrt{3}}. \tag{9}$$

Similarly,

$$\int_0^{\infty} \sin(xy)\eta(ix)\,dx = \sqrt{\frac{\pi}{2y}} \frac{\sinh\sqrt{8\pi y/3} - \sin\sqrt{8\pi y/3}}{\cosh\sqrt{8\pi y/3} + \cos\sqrt{8\pi y/3}}. \tag{10}$$

By dividing both sides of (10) by y and integrating over y from 0 to ∞, we obtain

$$\int_0^{\infty} \frac{dx}{x^2} \frac{\sinh(x) - \sin(x)}{\cosh(x) + \cos(x)} = \frac{\pi}{4} \tag{11}$$

an integral of the type that led to this study.

Next, Jacobi's triple identity [7] may be written

$$\eta^3(ix) = \sum_{n=0}^{\infty} (-1)^n(2n+1) q^{(2n+1)^2/8}. \tag{12}$$

By multiplying both sides by q^{z-1} and integrating over $0<q<1$ as before, we find, with $q = e^{-2\pi x}$,

$$\int_0^1 \frac{dq}{q} q^z \eta^3(ix) = 8\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2+8z} = \frac{2\pi}{\cosh(\pi\sqrt{2z})}, \tag{13}$$

which gives us the Laplace transform

$$\int_0^\infty e^{-xy}\eta^3(ix)\,dx = \operatorname{sech}\sqrt{\pi y}. \tag{14}$$

As for (6), if $F(t)$ denotes the inverse Laplace transform of $f(x)$, we have the identity

$$\int_0^\infty f(x)\eta^3(ix)\,dx = \int_0^\infty \frac{F(t)}{\cosh\sqrt{\pi t}}dt. \tag{15}$$

Formula (15) is much more flexible than (6) and leads to a variety of interesting looking integrals, a number of which are displayed in Appendix A, including the mysterious (A.7)

$$\int_0^\infty \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}}e^{-\pi x/4}\prod_{n=0}^\infty\left(1-e^{-2\pi nx}\right)^3 dx = \sqrt{2}-1. \tag{16}$$

This is derived by noting that the Laplace transform of $F(t) = \sin(at)/\sqrt{\pi t}$ is the imaginary part of $(a+ix)^{-1/2}$. Inserting this into (15) and using Parseval's identity for the cosine Fourier transform to simplify the integral on the right-hand side, one obtains (16) for $a = 1$.

By proceeding analogously to the derivation of (11), we find a second example of the "Glaisher–Ramanujan" class

$$\int_0^\infty \frac{dx}{x}\frac{\sinh(x/2)\sin(x/2)}{\cosh(x)+\cos(x)} = \frac{\pi}{8}. \tag{17}$$

In conclusion, we have opened a way to produce many integrals over the Dedekind-η function and evaluated one or two Glaisher–Ramanujan integrals. There exist many further q-identities similar to (3) and (12) which might prove productive for extending this investigation.

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Appendix A

$$\int_0^\infty e^{-xy}\eta^3(ix)\,dx = \operatorname{sech}\sqrt{\pi y}, \tag{A.1}$$

$$\int_0^\infty \frac{\eta^3(ix)}{x+a}\,dx = \frac{2}{\pi}\int_0^\infty \frac{xe^{-ax^2/\pi}}{\cosh(x)}\,dx, \tag{A.2}$$

$$\int_0^\infty x^{-\nu}\eta^3(ix)\,dx = \frac{4}{\pi^\nu}\frac{\Gamma(2\nu)}{\Gamma(\nu)}\beta(2\nu) \quad (\nu > 0), \tag{A.3}$$

$$\int_0^\infty \frac{\eta^3(ix)}{\sqrt{x+a}}\,dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-ax^2/\pi}}{\cosh(x)}\,dx, \tag{A.4}$$

$$\int_0^\infty x^{-1/2}e^{-a/x}\eta^3(ix)\,dx = \operatorname{sech}\sqrt{\pi a}, \tag{A.5}$$

$$\int_0^\infty e^{-xy}\eta^3(ix)\frac{dx}{x} = \frac{2}{\pi}\int_{\sqrt{\pi y}}^\infty x\operatorname{sech}(x)\,dx, \tag{A.6}$$

$$\int_0^\infty \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}}\eta^3(ix)\,dx = \sqrt{2}-1, \tag{A.7}$$

$$\int_0^\infty x^{-1/2}\cos(a/x)\eta^3(ix)\,dx = 2\frac{\cos\sqrt{\pi a/2}\cosh\sqrt{\pi a/2}}{\cos\sqrt{2\pi a}+\cosh\sqrt{2\pi a}}, \tag{A.8}$$

$$\int_0^\infty x^{-1/2}\operatorname{erf}(\sqrt{bx})\eta^3(ix)\,dx = \frac{4}{\pi}\arctan\left(\tanh\frac{1}{2}\sqrt{\pi b}\right), \tag{A.9}$$

$$\int_0^\infty x^{-1/2}e^{a/x}\operatorname{erfc}(\sqrt{a/x})\eta^3(ix)\,dx = \frac{1}{\pi\sqrt{a}}\left[\psi\left(\frac{1}{2}\sqrt{a/\pi}+\frac{3}{4}\right)-\psi\left(\frac{1}{2}\sqrt{a/\pi}+\frac{1}{4}\right)\right], \tag{A.10}$$

$$\int_0^\infty \cos(xy)\eta^3(ix)\,dx = \frac{\cosh(\sqrt{\pi y/2})\cos(\sqrt{\pi y/2})}{\sinh^2(\sqrt{\pi y/2})+\cos^2(\sqrt{\pi y/2})}, \tag{A.11}$$

$$\int_0^\infty \sin(xy)\eta^3(ix)\,dx = \frac{\sinh(\sqrt{\pi y/2})\sin(\sqrt{\pi y/2})}{\sinh^2(\sqrt{\pi y/2})+\cos^2(\sqrt{\pi y/2})}, \tag{A.12}$$

$$\int_0^\infty \eta(ix)\,dx = \frac{2\pi}{\sqrt{3}}, \tag{A.13}$$

$$\int_0^\infty \eta^3(ix)\,dx = 1, \tag{A.14}$$

$$\int_0^\infty x^n\eta^3(ix)\,dx = \frac{4n!}{\pi^{n+1}}\beta(2n+1). \tag{A.15}$$

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